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QCD EFFECTIVE ACTIONS FROM THE SOLUTIONS OF THE TRANSPORT EQUATIONS

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We solve the collisionless transport equations of a quark-gluon plasma interacting through mean chromodynamic fields. The system is assumed to be translation invariant in one or more space-time directions. We present exact solutions that hold if the vector gauge fields in the direction of the translation invariance commute with their covariant derivatives. We also solve the equations perturbatively when the commutation condition is relaxed. Further, we derive the color current and the associated effective action. For the static quasi-equilibrium system, our results reproduce the full one-loop effective action of QCD in the presence of constant background fields, where the above mentioned commutation condition is satisfied.

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I. INTRODUCTION

When the temperature T of the quark-gluon plasma is much greater than the QCD scale parameter $\Lambda_{\rm QCD}$, the hard modes, i.e. those with momenta of the order of T or larger, are weakly interacting and they can be described within perturbative QCD [1]. The dynamics of the soft sector, however, remains non-perturbative even at arbitrarily large temperature [2], as signalled by severe infrared divergences [3]. Then, one has to refer to effective theories to get an insight into the soft mode dynamics. Such theories, see e.g. [4,5], can be derived from QCD by integrating out the hard modes, but constructing them is by far not a simple task. Consequently, one often relies on more or less heuristic approaches, usually exploiting a semi-classical or classical field approximation because the occupation numbers of the soft gluonic modes are large.

A very natural effective approach is provided by the kinetic theory, where the hard modes are treated as (quasi-)particles while the soft gluonic ones contribute to the chromodynamic mean field. The transport theory has been formulated in two versions. The first one treats the color degrees of freedom as a classical continuous variable which, as position or momentum, evolves in time. A starting point of the theory are the Wong equations [6], which describe a classical particle that interacts with the chromodynamic field due to the color charge. Then, one immediately gets the Liouville and the transport equations [7] of a many-body quark-gluon system. The physical content of the theory is rather transparent and numerous results, for example transport coefficients, can be easily obtained. Even the simplest collisionless transport equations, where the dissipation phenomena are neglected, provides a surprisingly rich dynamics. The transport theory with the classical color became really reliable when the theory was found [8] to reproduce the QCD hard-thermal-loop dynamics [4,9,10]. It was further established [11] that the theory supplemented by the collision terms, obtained integrating out soft fluctuations around the mean fields, agrees with the QCD effective approaches [5,12]. The relationship between the transport theory with classical color and QCD can be found through the study of the quantum path integral within a saddle-point approximation [13].

In the second version of the QCD transport theory [7,14], the color charges are represented, in full accordance with QCD, by a matrix structure of the distribution function. The Vlasov transport equation of quarks was derived [15] directly from QCD, by analyzing the motion of quantum quarks in the classical chromodynamic field. The gluon transport equation was found [16,17], by splitting the gluon field into the mean field and the contribution representing the particle excitations. It was further observed [18] that the quark and gluon transport equations are formally identical when the first one is written in the fundamental representation and the second one in the adjoint representation. Then, the quark and gluon distribution functions are $N_c \times N_c$ and $(N_c^2-1)\times(N_c^2-1)$ matrices, respectively, for the $SU(N_c)$ gauge group. The early development of the quark-gluon kinetic theory was summarized at ref. [19].

In quasi-equilibrium, the matrix transport theory was proved [20] to be fully equivalent to the QCD hard-loop approach [4,9,10]. The kinetic approach, which can be treated as a local representation of the non-local hard-loop action, is particularly useful to study the collective excitations of the quark-gluon plasma, see [21] for a review. More recently, the quasi-equilibrium kinetic equations have been derived beyond the collisionless limit [22] and the QCD effective theories [5] have again been correctly reproduced.

A natural question that arises is where the agreement between the kinetic theory, either with the classical color or in the matrix form, and the finite temperature diagrammatic approach breaks down. Surprisingly enough, the effective action of the static fields provided by the kinetic equations agrees with that obtained within perturbative QCD even at the g^3 order [23], where an operator responsible for C-odd processes appears for systems with finite baryon density. However, in a subsequent study [24] the kinetic theory with the classical color has been found to reproduce the g^4 contribution to the effective action only in the limit of high-dimensional color representations. Thus, the limitations of the classical approach have been explicitly determined.

The aim of this paper is to clarify whether the difficulties faced by the classical color transport theory can be overcome when the matrix formulation is used. We explore how far the limits of the non-Abelian kinetic approach can be extended. For this purpose, we look for the solutions of the transport equations for quarks and gluons interacting with a chromodynamic mean field. The system is assumed to be translation invariant in one or more space-time directions. Thus, our considerations hold, in particular, for static and for homogeneous systems. We first present exact solutions when the vector gauge fields in the direction of the translation invariance commute with their covariant derivatives. Then, we use perturbation theory to solve the transport equation when the commutation condition is relaxed. Once the solutions are known, we derive the corresponding color current, and then the associated effective action. In the case of thermodynamic equilibrium, our results agree with those obtained by computing the one-loop effective action of QCD in the presence of a constant background field [2,25,26], which also corresponds to the effective potential of the dimensionally reduced theory [27]. However, for static and non-constant background fields such that the commutation condition is not satisfied, we find additional non-local operators that correct the effective potential of [27], in what seems to be a discrepancy between the two approaches.

The paper is organized as follows. In Sec. II we review the transport theory approach we use in this paper. Exact solutions of the transport equations for translation invariant systems are discussed in Sec. III while in Sec. IV we solve the equations perturbatively. The effective actions for static and for homogeneous systems close to equilibrium are derived in Sec. V and we conclude our considerations in Sec. VI. The evaluation of some momentum integrals is left for Appendix A and we collect in Appendix B some formulas of the traces of $SU(N_c)$ generators in the adjoint representation.

II. TRANSPORT EQUATIONS

In this section we briefly review the transport theory of quarks and gluons [19,18]. While we will restrict the discussion to QCD, with N_f massless quarks and antiquarks carrying color in the fundamental representation and gluons in the adjoint, the results could easily be generalized to a different non-Abelian theory with different field content.

The distribution function of (anti-)quarks $Q(p,x)$ ($\bar{Q}(p,x)$) is a hermitian $N_c \times N_c$ matrix in color space (for a $SU(N_c)$ color group); x denotes the space-time quark coordinate and

p its momentum, which is not constrained by the mass-shell condition. The spin of quarks and gluons is taken into account as an internal degree of freedom. The distribution function transforms under a local gauge transformation M as

$$Q(p,x) \rightarrow M(x) Q(p,x) M^\dagger(x) \; . \tag{2.1}$$

Here and most cases below, the color indices are suppressed. The distribution function of hard gluons is a hermitian $(N_c^2 - 1) \times (N_c^2 - 1)$ matrix, which transforms as

$$G(p,x) \rightarrow \mathcal{M}(x) G(p,x) \mathcal{M}^\dagger(x) \; , \tag{2.2}$$

where

$$\mathcal{M}_{ab}(x) = \mathrm{Tr}[\tau_a M(x) \tau_b M^\dagger(x)] \; ,$$

with $\tau_a, \; a = 1, ..., N_c^2 - 1$ being the $SU(N_c)$ group generators in the fundamental representation with $\mathrm{Tr}(\tau_a \tau_b) = \frac{1}{2}\delta_{ab}$.

In a collisionless limit, the distribution functions of quarks and gluons satisfy the transport equations:

$$p^\mu D_\mu Q(p,x) + \frac{g}{2}\, p^\mu \left\{ F_{\mu\nu}(x), \frac{\partial Q(p,x)}{\partial p_\nu} \right\} = 0 \; , \tag{2.3a}$$

$$p^\mu D_\mu \bar{Q}(p,x) - \frac{g}{2}\, p^\mu \left\{ F_{\mu\nu}(x), \frac{\partial \bar{Q}(p,x)}{\partial p_\nu} \right\} = 0 \; , \tag{2.3b}$$

$$p^\mu \mathcal{D}_\mu G(p,x) + \frac{g}{2}\, p^\mu \left\{ \mathcal{F}_{\mu\nu}(x), \frac{\partial G(p,x)}{\partial p_\nu} \right\} = 0 \; , \tag{2.3c}$$

where g is the QCD coupling constant, $\{...,...\}$ denotes the anticommutator; the covariant derivatives D_μ and $\mathcal{D}_\mu$ act as

$$D_\mu = \partial_\mu - ig[A_\mu(x), ...\,] \; , \qquad \mathcal{D}_\mu = \partial_\mu - ig[\mathcal{A}_\mu(x), ...\,] \; ,$$

A_μ and $\mathcal{A}_\mu$ being four-potentials in the fundamental and adjoint representations, respectively:

$$A^\mu(x) = A^\mu_a(x) \tau_a \; , \qquad \mathcal{A}^\mu_{ab}(x) = -i f_{abc} A^\mu_c(x) \; ,$$

and f_{abc} are the structure constants of the $SU(N_c)$ group. Since the generators of $SU(N_c)$ in the adjoint representation are given by $(T_a)_{bc} = -if_{abc}$, one can also write $\mathcal{A}^\mu = A^\mu_a T^a$. The stress tensor in the fundamental representation is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$, while $\mathcal{F}_{\mu\nu}$ denotes the field strength tensor in the adjoint representation.

Sometimes it is convenient to project the matrix equations (2.3) into their colorless and colored components. For the quark distribution function we define

$$Q(p,x) = \tilde{q}(p,x) + q^a(p,x) \tau^a \; . \tag{2.4}$$

Then, we can deduce from Eq. (2.3a) a set of coupled equations for the different components of Q defined by Eq. (2.4). More precisely, we find

$$p^\mu \partial_\mu \tilde{q}(p,x) + \frac{g}{2N_c}\, p^\mu F^a_{\mu\nu}(x) \frac{\partial q^a(p,x)}{\partial p_\nu} = 0 \,, \tag{2.5a}$$

$$p^\mu D^{ab}_\mu q^b(p,x) + \frac{g}{2}\, d^{abc} F^b_{\mu\nu}(x) \frac{\partial q^c(p,x)}{\partial p_\nu} + g\, p^\mu F^a_{\mu\nu}(x) \frac{\partial \tilde{q}(p,x)}{\partial p_\nu} = 0 \,, \tag{2.5b}$$

where d^{abc} are the totally symmetric structure constants of $SU(N_c)$ and $D^{ac}_\mu = \partial_\mu \delta^{ac} + g f^{abc} A^b_\mu$. Similar equations can be written for the antiquark and gluon distribution functions. Eqs. (2.5) reflect in a very clear way that transport phenomena of colorless and colored fluctuations are coupled beyond the lowest order in the gauge coupling constant. Eqs. (2.5) might be very useful when collisions are also taken into account. In this study, however, we find more convenient to work with the matrix equations (2.3).

Once the solution of the transport equations is known, we can obtain the color current associated to the plasma constituents. The color current is expressed in the fundamental representation as

$$j^\mu(x) = -\frac{g}{2} \int dP\, p^\mu \Big[Q(p,x) - \bar{Q}(p,x) - \frac{1}{N_c} \mathrm{Tr}[Q(p,x) - \bar{Q}(p,x)] + 2i\tau_a f_{abc} G_{bc}(p,x) \Big] \,, \tag{2.6}$$

so that $j^\mu_a(x) = 2\mathrm{Tr}(\tau_a j^\mu(x))$, and the momentum measure

$$dP = \frac{d^4p}{(2\pi)^3} 2\Theta(p_0)\, \delta(p^2) \tag{2.7}$$

takes into account the mass-shell condition $p_0 = |\mathbf{p}|$. Throughout the paper, we neglect the quark masses, although those might easily be taken into account by modifying the mass-shell constraint in the momentum measure. A sum over helicities, two per particle, and over quark flavors N_f is understood in Eq. (2.6), even though it is not explicitly written down.

In the transport theory framework one can consider two different physical situations: 1) the gauge fields entering into the transport equations (2.3) are external, not due to the plasma constituents; 2) the gauge fields can be generated self-consistently by the quarks and gluons. In the latter case, one also has to solve the Yang-Mills equation

$$D_\mu F^{\mu\nu}(x) = j^\nu(x) \,, \tag{2.8}$$

where the color current is given by Eq. (2.6).

The color current can be derived from an effective action added to the Yang-Mills one. By means of the relation

$$j^\mu_a = - \frac{\delta S}{\delta A^a_\mu} \,, \tag{2.9}$$

where $S \equiv \int d^4x\, \mathcal{L}$, one can obtain the effective action, up to an integration constant, from the knowledge of the color current. In the remaining part of this article we will use this approach to obtain the effective action in different physical situations.

III. EXACT SOLUTIONS

Finding exact solutions of the transport equations (2.3) is in general a difficult task. However, it is possible to find such solutions under some restrictive conditions. Here we consider a system where both the vector gauge field and the distribution functions are invariant under the space-time translation(s), i.e.

$$\partial_{\alpha_i} A^\mu(x) = 0 \; , \qquad \mu = 0,1,2,3 \; , \tag{3.1}$$

and

$$\partial_{\alpha_i} Q(p,x) = \partial_{\alpha_i} \bar{Q}(p,x) = \partial_{\alpha_i} G(p,x) = 0 \; , \tag{3.2}$$

for a fixed α_i, where α_i can involve more than one Lorentz index. For example, if $\alpha_i = 0$ the system is static while for $\alpha_i = 1,2,3$ the gauge field and the distribution functions depend only on time.

To solve Eqs. (2.3) for translation invariant systems along the direction x^{α_i}, we will take into account two known facts. First, in an electromagnetic plasma, an exact solution of the corresponding transport equation is given by any function of the canonical momentum $p_{\alpha_i} - eA_{\alpha_i}(x)$, if $\partial_{\alpha_i} A_\mu = 0$ [28]. Second, the non-Abelian transport equations of particles carrying a classical color charge I^a for these translation invariant systems are also solved by any function of the canonical momentum $p_{\alpha_i} - gA^a_{\alpha_i}(x) I^a$, if $\partial_{\alpha_i} A^a_\mu = 0$ [24]. Because the first case represents the Abelian limit of the transport equations (2.3), while the second one corresponds to the limit of high-dimensional color representations[‡], one expects that the solutions of Eqs. (2.3) are of the form:

$$Q(p,x) = f(p_{\alpha_i} - gA_{\alpha_i}(x)) = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} A_{\alpha_1}(x)\, A_{\alpha_2}(x) \cdots A_{\alpha_n}(x)\, \frac{\partial^n f(p_{\alpha_i})}{\partial p_{\alpha_1}\, \partial p_{\alpha_2} \ldots \partial p_{\alpha_n}} \; , \tag{3.3a}$$

$$\bar{Q}(p,x) = \bar{f}(p_{\alpha_i} + gA_{\alpha_i}(x)) = \sum_{n=0}^{\infty} \frac{g^n}{n!} A_{\alpha_1}(x)\, A_{\alpha_2}(x) \cdots A_{\alpha_n}(x)\, \frac{\partial^n \bar{f}(p_{\alpha_i})}{\partial p_{\alpha_1}\, \partial p_{\alpha_2} \ldots \partial p_{\alpha_n}} \; , \tag{3.3b}$$

$$G(p,x) = f_g(p_{\alpha_i} - g\mathcal{A}_{\alpha_i}(x)) = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \mathcal{A}_{\alpha_1}(x)\, \mathcal{A}_{\alpha_2}(x) \cdots \mathcal{A}_{\alpha_n}(x)\, \frac{\partial^n f_g(p_{\alpha_i})}{\partial p_{\alpha_1}\, \partial p_{\alpha_2} \ldots \partial p_{\alpha_n}} \; , \tag{3.3c}$$

where it is understood that a sum is taken over the repeated indices. The functions f, $\bar{f}$ and f_g are, in principle, arbitrary but they can be fixed by additional considerations.

Let us note that the distribution functions given by Eqs. (3.3) transform covariantly, i.e. according to Eqs. (2.1) and (2.2), even though the potentials A^μ, $\mathcal{A}^\mu$, in general, do not. Indeed,

$$A^{\alpha_i}(x) \rightarrow M(x) A^{\alpha_i}(x) M^\dagger(x) - i(\partial^{\alpha_i} M(x)) M^\dagger(x) \; . \tag{3.4}$$

[‡]While we are considering here the transport equations only for particles carrying color in the fundamental and adjoint representations, the equations are expected to have the same structure in any other non-Abelian representation.

However, the second term in the r.h.s of Eq. (3.4), which transforms non-covariantly, is eliminated because of the condition (3.1).

Let us check under which conditions the ansatz Eq. (3.3) solves the transport equations. We first note that Eq. (3.3a) is totally symmetric under the exchange of indices $\alpha_1, \cdots, \alpha_n$. Inserting Eq. (3.3a) into the transport equation (2.3a), one finds

$$p^\mu D_\mu Q(p,x) = p^\mu \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \sum_{s=0}^{n-1} A_{\alpha_1} \cdots A_{\alpha_s} (D_\mu A_{\alpha_{s+1}}) \cdots A_{\alpha_n} \frac{\partial^n f(p_{\alpha_i})}{\partial p_{\alpha_1} \partial p_{\alpha_2} \ldots \partial p_{\alpha_n}} \ , \tag{3.5}$$

$$\frac{g}{2}\, p^\mu \left\{ F_{\mu\nu}(x), \frac{\partial Q(p,x)}{\partial p_\nu} \right\} = -p^\mu \sum_{n=0}^{\infty} \frac{(-g)^{n+1}}{2\, n!} \left\{ D_\mu A_{\alpha_i}, A_{\alpha_1} \cdots A_{\alpha_n} \right\} \frac{\partial^{n+1} f(p_{\alpha_i})}{\partial p_{\alpha_i} \partial p_{\alpha_1} \ldots \partial p_{\alpha_n}} \ , \tag{3.6}$$

where we have used the following property of a commutator

$$[X, Y^n] = \sum_{s=0}^{n-1} Y^s [X,Y] Y^{n-s-1}$$

to derive Eq. (3.5). We have also taken into account that $F_{\mu\alpha_i} = D_\mu A_{\alpha_i}$ because of Eq. (3.1).

It is easy to observe that the terms in Eq. (3.5) and Eq. (3.6) that correspond to the same order of the derivative of f cancel each other exactly if

$$[D_\mu A_{\alpha_i}, A_{\alpha_j}] = 0 \ , \qquad \mu = 0,1,2,3 \ , \tag{3.7}$$

where A_{α_j} is also in the direction of the translation invariance. The same condition is obtained for the antiquark distribution function, while for gluons one finds that Eq. (3.3c) is an exact solution of Eq. (2.3c) if

$$[\mathcal{D}_\mu \mathcal{A}_{\alpha_i}, \mathcal{A}_{\alpha_j}] = 0 \ , \qquad \mu = 0,1,2,3 \ . \tag{3.8}$$

However, it is easy to prove that Eq. (3.8) is automatically satisfied if Eq. (3.7) holds.

For static systems ($\alpha = 0$), Eq. (3.7) reduces to the commutation relation between the color electric field and A_0. In a more general situation, the condition Eq. (3.7) simplifies the non-Abelian field dynamics in the direction of the translation invariance. Note that the commutation condition is trivially satisfied in the Abelian limit. If T_R^a is a generator of a representation R of $SU(N_c)$ then after a normalization of these generators one would get $[T_R^a, T_R^b] \to 0$ for high-dimensional representations. Consequently, the commutation condition would also be satisfied. This explains how to reconcile the matrix results with those obtained with the non-Abelian transport equations for classical color.

Once the solution for the quark, antiquark and gluon distribution functions are known, one can compute the color current. Inserting (3.3) into Eq. (2.6), we get

$$\begin{aligned} j_a^\mu(x) = 2 \sum_{n=0}^{\infty} \frac{(-g)^{n+1}}{n!} A_{\alpha_1}^{c_1}(x) \cdots A_{\alpha_n}^{c_n}(x) \int dP\, p^\mu \Bigg[N_f \mathrm{Tr}[\tau_a \tau_{c_1} \cdots \tau_{c_n}] \Bigg[\frac{\partial^n f(p_{\alpha_i})}{\partial p_{\alpha_1} \cdots \partial p_{\alpha_n}} \\ + (-1)^{n+1} \frac{\partial^n \bar{f}(p_{\alpha_i})}{\partial p_{\alpha_1} \cdots \partial p_{\alpha_n}} \Bigg] + \mathrm{Tr}[T_a T_{c_1} \cdots T_{c_n}] \frac{\partial^n f_g(p_{\alpha_i})}{\partial p_{\alpha_1} \cdots \partial p_{\alpha_n}} \Bigg] \ . \end{aligned} \tag{3.9}$$

The factor 2 in the above equation arises from the two helicities associated with every particle species. When the functions f, $\bar{f}$ and f_g are determined, one can evaluate the momentum integral of Eq. (3.9), and then, after solving Eq. (2.9), one obtains the associated effective action.

IV. PERTURBATIVE SOLUTIONS

A. General Considerations

In Sec. III we have found exact solutions of the collisionless transport equations (2.3) for translation invariant systems that obey the extra condition Eq. (3.7). In this section we treat the transport equations perturbatively and find solutions that are not constrained by the condition Eq. (3.7).

We assume here that

$$g \ll 1 \ , \tag{4.1}$$

i.e. we deal with the weak coupling regime of the theory, and consider an expansion of the distribution function of the form

$$Q = Q^{(0)} + Q^{(1)} + Q^{(2)} + ... \tag{4.2}$$

where the 0-th term is a known function

$$Q^{(0)}(p,x) = f(p_{\alpha_i}) \ . \tag{4.3}$$

The higher-order terms are determined by the equation

$$p^\mu D_\mu Q^{(n)}(p,x) + \frac{g}{2}\, p^\mu \left\{ F_{\mu\nu}(x), \frac{\partial Q^{(n-1)}(p,x)}{\partial p_\nu} \right\} = 0 \ . \tag{4.4}$$

Of course, the same treatments should be followed to study the antiquark and gluon distribution functions, but those are nearly identical.

The iterative procedure based on Eq. (4.4) does not correspond to a strict expansion in powers of g. Such an expansion would force us to split all covariant derivatives into a derivative and a commutator part, resulting in the breaking of the gauge covariance of every term in the perturbative series. We maintain the gauge covariance of every term in Eq. (4.2) at the expense of reorganizing the perturbative expansion.

It should be noticed that for f being the equilibrium distribution function, the first term ($n=1$) of the above perturbation series reproduces the hard thermal loops of QCD [4,20]. Let us also stress that while we are applying here a perturbative method to the transport equations in their matrix form, fully equivalent results can be obtained using the projected equations (2.5). Such an approach has been carried out by Bödeker and Laine to order g^2 for static quasi-equilibrium systems [23]. We extend that analysis by pushing the perturbative procedure to higher orders in g. We do not attempt to solve the transport equations in full generality, as the solutions at every order then turn out to be highly non-local. We reduce our study to translation invariant systems when the solutions are much simplified.

B. From g to g^4 order

In this subsection, we consider a translation invariant system that obeys Eq. (3.1) for a fixed α_i. We also assume that the unperturbed distribution function depends only on p_{α_i} as in Eq. (4.3). The first-order correction to $Q^{(0)}$ is obtained by solving the equation

$$p^\mu D_\mu Q^{(1)} = -g\, p^\mu F_{\mu\alpha_1} \frac{\partial f(p_{\alpha_i})}{\partial p_{\alpha_1}} = -g p^\mu D_\mu A_{\alpha_1} \frac{\partial f(p_{\alpha_i})}{\partial p_{\alpha_1}} \, . \tag{4.5}$$

The solution reads (up to a function h, such that $p^\mu D_\mu h = 0$, that we will neglect throughout)

$$Q^{(1)}(p,x) = -g\, A_{\alpha_1}(x) \frac{\partial f(p_{\alpha_i})}{\partial p_{\alpha_1}} \, , \tag{4.6}$$

and it coincides with the first term of the ansatz (3.3a).

The transport equation at second-order is

$$p^\mu D_\mu Q^{(2)} = \frac{g^2}{2} \left\{ p^\mu D_\mu A_{\alpha_1}, A_{\alpha_2} \right\} \frac{\partial^2 f(p_{\alpha_i})}{\partial p_{\alpha_1} \partial p_{\alpha_2}} \, , \tag{4.7}$$

but it can be rewritten as

$$p^\mu D_\mu Q^{(2)} = \frac{g^2}{2}\, p^\mu D_\mu \left(A_{\alpha_1} A_{\alpha_2} \right) \frac{\partial^2 f(p_{\alpha_i})}{\partial p_{\alpha_1} \partial p_{\alpha_2}} \, . \tag{4.8}$$

Thus, the second-order solution is

$$Q^{(2)}(p,x) = \frac{g^2}{2}\, A_{\alpha_1}(x) A_{\alpha_2}(x) \frac{\partial^2 f(p_{\alpha_i})}{\partial p_{\alpha_1} \partial p_{\alpha_2}} \, , \tag{4.9}$$

which again coincides with the respective term of the ansatz (3.3a).

The third-order equation is

$$p^\mu D_\mu Q^{(3)} = -\frac{g^3}{4} \left\{ p^\mu D_\mu A_{\alpha_1}, A_{\alpha_2} A_{\alpha_3} \right\} \frac{\partial^3 f(p_{\alpha_i})}{\partial p_{\alpha_1} \partial p_{\alpha_2} \partial p_{\alpha_3}} \, , \tag{4.10}$$

and its r.h.s. is *not* proportional to a covariant derivative of a third power of A_{α_i}. However, the anticommutator of the r.h.s. of the equation can be rewritten as follows:

$$p^\mu D_\mu Q^{(3)} = -\frac{g^3}{3!} \left(p^\mu D_\mu \left(A_{\alpha_1} A_{\alpha_2} A_{\alpha_3} \right) + \frac{1}{2} \Big[[p^\mu D_\mu A_{\alpha_1}, A_{\alpha_2}], A_{\alpha_3} \Big] \right) \frac{\partial^3 f(p_{\alpha_i})}{\partial p_{\alpha_1} \partial p_{\alpha_2} \partial p_{\alpha_3}} \, . \tag{4.11}$$

The term with the commutator cannot be expressed as a total covariant derivative. Thus, the solution of the above equation contains two pieces: a term that is local in the gauge fields and a non-local part. Namely,

$$Q^{(3)}(x,p) = -\frac{g^3}{3!} A_{\alpha_1}(x) A_{\alpha_2}(x) A_{\alpha_3}(x) \frac{\partial^3 f(p_{\alpha_i})}{\partial p_{\alpha_1} \partial p_{\alpha_2} \partial p_{\alpha_3}} + Q^{(3)}_{\rm nl}(x,p) \tag{4.12}$$

with the non-local term obeying the equation

$$p^\mu D_\mu Q_{\rm nl}^{(3)} = -\frac{g^3}{12}\Big[[p^\mu D_\mu A_{\alpha_1}, A_{\alpha_2}], A_{\alpha_3}\Big]\frac{\partial^3 f(p_{\alpha_i})}{\partial p_{\alpha_1}\partial p_{\alpha_2}\partial p_{\alpha_3}} \ . \tag{4.13}$$

One observes that when the condition Eq. (3.7) is satisfied, $Q_{\rm nl}^{(3)} = 0$ and we recover the solution of order g^3 of Eq. (3.3a). But this term is non-zero under more general circumstances. We also notice that if

$$[\partial_\mu A_{\alpha_i}, A_{\alpha_j}] = 0 \ , \tag{4.14}$$

the non-local piece is proportional to g^4.

Let us solve Eq. (4.13). Using the $SU(N_c)$ algebra, we rewrite the commutator as

$$p^\mu D_\mu Q_{\rm nl}^{(3)} = -\frac{g^3}{12} f^{acd} f^{deb} \tau^a (p^\mu D_\mu A_{\alpha_1})^e A^c_{\alpha_2} A^b_{\alpha_3} \frac{\partial^3 f(p_{\alpha_i})}{\partial p_{\alpha_1}\partial p_{\alpha_2}\partial p_{\alpha_3}} \equiv R^a \, \tau^a \ . \tag{4.15}$$

Consequently, $Q_{\rm nl}^{(3)} = Q_{\rm nl}^{(3)a}\tau^a$ with

$$p^\mu D_\mu^{ac} Q_{\rm nl}^{(3)c} = R^a \ . \tag{4.16}$$

The solution can be expressed as

$$Q_{\rm nl}^{(3)a}(p,x) = \int d^4y \, \langle x | \frac{1}{p\cdot D} | y \rangle_{ab} \, R^b(y) \ , \tag{4.17}$$

where $1/p\cdot D$ is the retarded Green function associated to the differential equation (4.16). The explicit form of $1/p\cdot D$ can be found, for example, in Sec. II of [29].

The fourth-order equation reads

$$p^\mu D_\mu Q^{(4)} = \frac{g^4}{12}\Big\{p^\mu D_\mu A_{\alpha_1}, A_{\alpha_2}A_{\alpha_3}A_{\alpha_4}\Big\} \frac{\partial^4 f(p_{\alpha_i})}{\partial p_{\alpha_1}\partial p_{\alpha_2}\partial p_{\alpha_3}\partial p_{\alpha_4}} + \frac{g^4}{24}\Big\{F_{\mu\nu}, \frac{\partial Q_{\rm nl}^{(3)}}{\partial p_\nu}\Big\} \ . \tag{4.18}$$

The first anticommutator in the r.h.s. of the equation can also be rewritten as a total covariant derivative plus an additional commutator. Thus,

$$\begin{aligned} p^\mu D_\mu Q^{(4)} = \frac{g^4}{4!}\Big(p^\mu D_\mu \left(A_{\alpha_1}A_{\alpha_2}A_{\alpha_3}A_{\alpha_4}\right) + \Big[[p^\mu D_\mu A_{\alpha_1}, A_{\alpha_2}], A_{\alpha_3}A_{\alpha_4}\Big]\Big) \frac{\partial^4 f(p_{\alpha_i})}{\partial p_{\alpha_1}\partial p_{\alpha_2}\partial p_{\alpha_3}\partial p_{\alpha_4}} \\ + \frac{g^4}{4!}\Big\{F_{\mu\nu}, \frac{\partial Q_{\rm nl}^{(3)}}{\partial p_\nu}\Big\} \ . \end{aligned} \tag{4.19}$$

We see that the solution is of the form

$$Q^{(4)}(x,p) = \frac{g^4}{4!} A_{\alpha_1}(x)A_{\alpha_2}(x)A_{\alpha_3}(x)A_{\alpha_4}(x) \frac{\partial^4 f(p_{\alpha_i})}{\partial p_{\alpha_1}\partial p_{\alpha_2}\partial p_{\alpha_3}\partial p_{\alpha_4}} + Q_{\rm nl}^{(4)}(x,p) \ , \tag{4.20}$$

where $Q_{\rm nl}^{(4)}$ represents the non-local contribution that vanishes if the condition Eq. (3.7) is satisfied. We note that the local piece coincides with the term g^4 of the ansatz (3.3a).

In principle, we could solve the equation for $Q^{(4)}_{\rm nl}$ and for the following terms of the perturbative expansion, finding that every term contains both local and non-local pieces in the gauge fields. However, we stop our analysis here as the structure of the non-local terms becomes more and more complex.

Adding the contributions of quarks, antiquarks and gluons, we can obtain the color current. Up to the term $n = 3$, the color current still has a relatively simple form and is given by

$$j^\mu_a(x) = 2\sum_{n=0}^{3} \frac{(-g)^{n+1}}{n!} A^{c_1}_{\alpha_1}(x)\cdots A^{c_n}_{\alpha_n}(x) \int dP\, p^\mu \Bigg[N_f {\rm Tr}[\tau_a \tau_{c_1}\cdots\tau_{c_n}] \Big[\frac{\partial^n f(p_{\alpha_i})}{\partial p_{\alpha_1}\cdots\partial p_{\alpha_n}} \tag{4.21}$$

$$+ (-1)^{n+1} \frac{\partial^n \bar f(p_{\alpha_i})}{\partial p_{\alpha_1}\cdots\partial p_{\alpha_n}} \Big] + {\rm Tr}[T_a T_{c_1}\cdots T_{c_n}] \frac{\partial^n f_g(p_{\alpha_i})}{\partial p_{\alpha_1}\cdots\partial p_{\alpha_n}} \Bigg] + j^\mu_{a,\rm nl}(x) \ ,$$

where the non-local term reads

$$j^\mu_{a,\rm nl}(x) = \frac{g^4}{6} \int dP \int d^4y\, \langle x| \frac{p^\mu}{p\cdot D} |y\rangle_{ab} \Bigg[2N_f {\rm Tr}\left(\tau^b \Big[[p\cdot DA_{\alpha_1}, A_{\alpha_2}], A_{\alpha_3}\Big]\right) \Big[\frac{\partial^3 f(p_{\alpha_i})}{\partial p_{\alpha_1}\partial p_{\alpha_2}\partial p_{\alpha_3}}$$

$$+ \frac{\partial^3 \bar f(p_{\alpha_i})}{\partial p_{\alpha_1}\partial p_{\alpha_2}\partial p_{\alpha_3}} \Big] + \frac{1}{N_c} {\rm Tr}\left(T^b \Big[[p\cdot \mathcal{D}\mathcal{A}_{\alpha_1}, \mathcal{A}_{\alpha_2}], \mathcal{A}_{\alpha_3}\Big]\right) \frac{\partial^3 f_g(p_{\alpha_i})}{\partial p_{\alpha_1}\partial p_{\alpha_2}\partial p_{\alpha_3}} \Bigg] \ . \tag{4.22}$$

In the following section we will get the effective action for static systems close to equilibrium. In that case, it is easy to see that for $n \geq 4$ the local pieces do not contribute to the current or effective action, as either the momentum integrals (Appendix A) or the traces of generators in the adjoint representation (Appendix B) vanish. However, the same does not hold true for the non-local terms.

V. EFFECTIVE ACTION OF QUASI-EQUILIBRIUM SYSTEMS

In this section we find the effective action for systems close to thermal equilibrium. Then, the functions f, $\bar f$ and f_g are of the Fermi-Dirac or Bose-Einstein form

$$f_{\rm FD}(E) = \frac{1}{e^{\beta(E-\mu)}+1} \ , \qquad \bar f_{\rm FD}(E) = \frac{1}{e^{\beta(E+\mu)}+1} \ , \qquad f_{\rm BE}(E) = \frac{1}{e^{\beta E}-1} \ , \tag{5.1}$$

where $\beta \equiv 1/T$, and T is the temperature, and μ is the quark chemical potential.

A. Static Systems

We first consider static systems satisfying the condition

$$[D_\mu A_0, A_0] = 0 \ , \qquad \mu = 0,1,2,3 \ . \tag{5.2}$$

The color current is given by Eq. (3.9) with $\alpha_i = 0$ while the distribution functions are those in Eq. (5.1).

After performing the momentum integral, one observes that $j^i_a = 0$ since

$$\int \frac{d\Omega_{\mathbf{v}}}{4\pi}\, v^i = 0 \ , \tag{5.3}$$

where $v^i = p^i/|\mathbf{p}|$ is the particle velocity, and the integral is performed over angular directions of $\mathbf{v}$. Thus, only j^0_a, i.e. the color current density, is non-vanishing. In the case of static systems, one easily finds the Lagrangian density by integrating Eq. (2.9). The fermionic contribution arises from

$$\mathcal{L}_f = -\frac{N_f}{\pi^2}\sum_{n=0}^{\infty}\frac{(-g)^{n+1}}{(n+1)!}\int_0^\infty dE\,E^2\left[\frac{d^n f_{\rm FD}(E)}{dE^n} + (-1)^{n+1}\frac{d^n \bar{f}_{\rm FD}(E)}{dE^n}\right]{\rm Tr}[A_0^{n+1}(\mathbf{x})] \ , \tag{5.4}$$

while that of gluons is

$$\mathcal{L}_g = -\frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{(-g)^{n+1}}{(n+1)!}\int_0^\infty dE\,E^2\,\frac{d^n f_{\rm BE}(E)}{dE^n}\,{\rm Tr}[\mathcal{A}_0^{n+1}(\mathbf{x})] \ . \tag{5.5}$$

In the above expressions we have made use of the mass-shell condition, which gives $E \equiv p_0 = |\mathbf{p}|$.

The momentum integrals of Eqs. (5.4) and (5.5) are evaluated in Appendix A. The fermionic integrals vanish for $n \geq 4$, and thus the series terminates at order g^4. The evaluation of the bosonic integral requires regularization for $n \geq 2$ (see Appendix A). In contrast to the fermionic integrals, they do not vanish for $n \geq 4$ if n is an even number. We note, however, that in Eq. (5.5) one has to evaluate the totally symmetric trace of $(n+1)$ adjoint generators for the term of order n, and the trace vanishes if $(n+1)$ is odd (see Appendix B). Thus, the series in Eq. (5.5) terminates also at the order g^4.

With the values of the momentum integrals given in Appendix A, we find

$$\frac{\mathcal{L}_f}{N_f} = g\,\frac{\mu}{3}\Big(T^2 + \frac{\mu^2}{\pi^2}\Big){\rm Tr}A_0 + \frac{g^2}{2}\Big(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\Big){\rm Tr}A_0^2 + \mu\frac{g^3}{3\pi^2}{\rm Tr}A_0^3 + \frac{g^4}{12\pi^2}{\rm Tr}A_0^4 \ , \tag{5.6}$$

where we have kept a linear term in the gauge potential that only survives in the Abelian limit where ${\rm Tr}A_0 = A_0$. For the gluons, we find

$$\mathcal{L}_g = \frac{g^2T^2}{6}{\rm Tr}\mathcal{A}_0^2 - \frac{g^4}{24\pi^2}{\rm Tr}\mathcal{A}_0^4 . \tag{5.7}$$

As shown in Appendix B, the traces of the fields in the adjoint representation can be expressed through the fundamental representation traces.

Let us also note that the above Lagrangians can be written as

$$\mathcal{L}_f = \frac{TN_f}{\pi^2}\,{\rm Tr}\int_0^\infty dE\,E^2\left[\ln(1+e^{-\beta(E-\mu-gA^0)}) + \ln(1+e^{-\beta(E+\mu+gA^0)})\right] \ , \tag{5.8}$$

$$\mathcal{L}_g = \frac{T}{\pi^2}\,{\rm Tr}\int_0^\infty dE\,E^2\,\ln(1-e^{-\beta(E-g\mathcal{A}^0)}) \ , \tag{5.9}$$

because

$$\frac{d}{dx}\ln(1\pm e^{-x}) = \pm\frac{1}{e^x \pm 1} \ .$$

Going to Euclidean time, we observe that the above results agree with the dimensionally reduced effective action [27], or with the full one-loop contribution to the effective potential for the phase of the Polyakov line [2,25,26]. We note that in the last computations, one is considering a static system in a constant background field A_0, which can always be chosen in diagonal form [2], and thus satisfies the condition Eq. (5.2).

Now, let us briefly consider a static system that does not satisfy Eq. (5.2). Using the solutions of Sec. IV B, we find that up to the order g^4 Eqs. (5.6) and (5.7) still hold, but at order g^4 and beyond there are corrections due to the non-local terms. The first non-local contribution appears at order g^4 from the non-local current Eq. (4.22) with $\alpha_i = 0$. To get the corresponding term in the effective action, one should still solve Eq. (2.9). We have not found the solution, but we also see no reason why the term should vanish. Thus, we conclude that the presence of the non-local contribution signals a discrepancy with the dimensionally reduced effective action [27] at order g^4, whenever the condition Eq. (5.2) is not satisfied.

B. Homogeneous Systems

We consider here time-dependent homogeneous systems that obey the condition Eq. (3.7) with $\alpha_i = 1, 2, 3$. For a system close to equilibrium, where the rotational symmetry is not broken, the solution of the transport equations can only depend on the modulus of the canonical three-momentum $|\mathbf{p} - g\mathbf{A}(t)|$. The color current in this situation is given by Eq. (3.9) with $f(p_{\alpha_i}), \bar{f}(p_{\alpha_i})$ and $f_g(p_{\alpha_i})$ replaced by $f_{\rm FD}(|\mathbf{p}|), \bar{f}_{\rm FD}(|\mathbf{p}|)$, and $f_{\rm BE}(|\mathbf{p}|)$, respectively.

The color current is then given by Eq. (3.9), but now $\alpha_i = 1, 2, 3$. To get the expression for the color current, we first need to perform the momentum integral of Eq. (3.9). The partial derivatives appearing in Eq. (3.9) can be explicitly evaluated. Using the chain rule, they can be expressed as partial derivatives of $E = |\mathbf{p}|$. Therefore,

$$\frac{\partial f}{\partial p_i} = \frac{\partial E}{\partial p_i}\frac{df}{dE} , \qquad \frac{\partial^2 f}{\partial p_i \partial p_j} = \frac{\partial^2 E}{\partial p_i \partial p_j}\frac{df}{dE} + \frac{\partial E}{\partial p_i}\frac{\partial E}{\partial p_j}\frac{d^2 f}{dE^2} ,$$

$$\begin{aligned} \frac{\partial^3 f}{\partial p_i \partial p_j \partial p_k} = & \frac{\partial^3 E}{\partial p_i \partial p_j \partial p_k}\frac{df}{dE} + \left(\frac{\partial E}{\partial p_i}\frac{\partial^2 E}{\partial p_j \partial p_k} + \frac{\partial E}{\partial p_j}\frac{\partial^2 E}{\partial p_i \partial p_k} + \frac{\partial E}{\partial p_k}\frac{\partial^2 E}{\partial p_i \partial p_j} \right)\frac{d^2 f}{dE^2} \\ & + \frac{\partial E}{\partial p_i}\frac{\partial E}{\partial p_j}\frac{\partial E}{\partial p_k}\frac{d^3 f}{dE^3} , \qquad \text{etc} , \end{aligned}$$

where $i, j, k = 1, 2, 3$. The partial derivatives of E can be written as functions of the particle velocities. Let us define

$$L_{(n)}^{i_1 \cdots i_n} \equiv \frac{\partial^n E}{\partial p_{i_1} \cdots \partial p_{i_n}} . \tag{5.10}$$

An explicit evaluation of the first terms gives

$$L_{(1)}^i = v^i , \qquad L_{(2)}^{ij} = \frac{1}{E}\left(\delta^{ij} - v^i v^j\right) ,$$

$$L_{(3)}^{ijk} = -\frac{1}{E}\left(L_{(2)}^{ij}v^k + L_{(2)}^{ik}v^j + L_{(2)}^{jk}v^i\right) \ , \qquad \text{etc.}$$

The angular integral of every term in the series of Eq. (3.9) can be easily calculated. Since the integrals of an even number of v^i vanish we only need:

$$\int \frac{d\Omega_{\mathbf{v}}}{4\pi}\, v^i v^j = \frac{1}{3}\,\delta^{ij} \ , \tag{5.11}$$

$$\int \frac{d\Omega_{\mathbf{v}}}{4\pi}\, v^i v^j v^k v^l = \frac{1}{15}\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right) \ . \tag{5.12}$$

The integrals over E are the same as those used in the previous subsection (see Appendix A).

The evaluation of the momentum integrals gives, as expected, $j_a^0 = 0$. The space components of the color current are, of course, non-zero with the quark and gluon contributions given, respectively, as

$$\begin{aligned} j_a^j &= \frac{g^2}{3}\Big(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\Big)\mathrm{Tr}(\tau_a A^j) \\ &+ \frac{g^4}{45\pi^2}\Big(\mathrm{Tr}(\tau_a A^j A^i A^i) + \mathrm{Tr}(\tau_a A^i A^j A^i) + \mathrm{Tr}(\tau_a A^i A^i A^j)\Big) \ , \end{aligned} \tag{5.13}$$

$$j_a^j = \frac{g^2T^2}{9}\mathrm{Tr}(T_a\mathcal{A}) + \frac{g^4}{90\pi^2}\Big(\mathrm{Tr}(T_a\mathcal{A}^j\mathcal{A}^i\mathcal{A}^i) + \mathrm{Tr}(T_a\mathcal{A}^i\mathcal{A}^j\mathcal{A}^i) + \mathrm{Tr}(T_a\mathcal{A}^i\mathcal{A}^i\mathcal{A}^j)\Big) \ . \tag{5.14}$$

These currents arise from the following Lagrangian densities:

$$\frac{\mathcal{L}_f}{N_f} = -\frac{g^2}{6}\Big(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\Big)\mathrm{Tr}(A_jA_j) - \frac{g^4}{90\pi^2}\left(\mathrm{Tr}(A_jA_jA_iA_i) + \frac{1}{2}\mathrm{Tr}(A_jA_iA_jA_i)\right) \ , \tag{5.15}$$

and

$$\mathcal{L}_g = -\frac{g^2T^2}{18}\mathrm{Tr}(\mathcal{A}_j\mathcal{A}_j) - \frac{g^4}{180\pi^2}\left(\mathrm{Tr}(\mathcal{A}_j\mathcal{A}_j\mathcal{A}_i\mathcal{A}_i) + \frac{1}{2}\mathrm{Tr}(\mathcal{A}_i\mathcal{A}_j\mathcal{A}_i\mathcal{A}_j)\right) \ . \tag{5.16}$$

We should point out that the terms proportional to g^2 in Eq. (5.15) and Eq. (5.16) agree with the hard thermal loop Lagrangians in the homogeneous limit [4]. If the condition Eq. (3.7) is not satisfied, the color current, and thus the associated effective Lagrangians, are corrected at order g^4 and beyond by the addition of the non-local terms, exactly as happened for the static systems. However, we will not write down these explicit terms here.

VI. CONCLUSIONS

Our results show the efficiency of transport theory in describing the quark-gluon plasma at soft scales. We have shown how the solutions of the collisionless transport equations for the static systems close to equilibrium reproduce the one-loop effective potential for the phase of the Polyakov line. Up to now, the results of the transport theory and quantum field theoretical computations have been known to agree only for the lower-dimensional operators, but our computations indicate that the agreement extends to the full one-loop effective action. We find a complete match of the transport results with those of the one-loop effective potential in the presence of a constant background field. For non-constant static background fields, transport theory predicts the appearance of non-local operators in the effective action, starting at order g^4 and beyond. This result then suggests a discrepancy with the dimensionally reduced effective theories [27], where these non-local operators are not present.

We have limited our analysis to translation invariant systems, when the solutions of the transport equations are local in the lower orders of the perturbative expansion. It would be desirable to solve the equations in full generality. However, the solutions are then complex non-local functions of the gauge fields, and their structure beyond order g^2 is not particularly enlightening. It is presumably more promising to explore the combined set of equations (2.3) and (2.8) numerically, thus allowing for a non-perturbative study of dynamical phenomena at soft scales, even beyond the hard thermal loop approximation.

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APPENDIX A: MOMENTUM INTEGRALS

1. Bosons

The bosonic integrals to be evaluated are of the form:

$$I_n^b = T^{3-n} \int_0^\infty dx\, x^2 \, \frac{d^n f_{\rm BE}(x)}{dx^n} \ . \tag{A1}$$

Expanding the Bose-Einstein distribution as

$$f_{\rm BE}(x) = \frac{1}{e^x - 1} = \sum_{m=1}^{\infty} e^{-mx}$$

and interchanging the order of summation and integration, the integral that has to performed reduces to

$$\int_0^\infty dx\, x^2 e^{-mx} = \frac{2}{m^3} \ . \tag{A2}$$

Therefore,

$$I_n^b = 2(-1)^n T^{3-n} \sum_{m=1}^{\infty} \frac{1}{m^{3-n}} = 2(-1)^n T^{3-n} \zeta(3-n) \ , \tag{A3}$$

where the solution is expressed in terms of the zeta function

$$\zeta(s) \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{1}{m^s} \ , \qquad \text{Re}s > 1 \ . \tag{A4}$$

While the integral (A1) is convergent, our final expression (A3) with the zeta function defined by Eq. (A4) is divergent, or rather ill-defined, for $n \geq 2$ because of the 'illegal' interchanging of the order of summation and integration. As is well known, the problem is resolved when the analytic continuation of $\zeta(s)$ instead of the definition (A4) is used. This is the so-called zeta function regularization procedure [30,31]. Then,

$$\begin{aligned}
\zeta(3) &\cong 1.202 \ , & \zeta(2) &= \frac{\pi^2}{6} \ , \\
\zeta(1) &= \infty \ , & \zeta(0) &= -\frac{1}{2} \ , \\
\zeta(-1) &= -\frac{1}{12} \ , & \zeta(-2) &= 0 \ , \\
\zeta(-3) &= \frac{1}{120} \ , & \zeta(-4) &= 0 \ , \\
\zeta(1-2k) &= -\frac{B_{2k}}{2k} \ , & k &= 1,2,3,... \\
\zeta(-2k) &= 0 \ , & k &= 1,2,3,...
\end{aligned} \tag{A5}$$

where B_l are the Bernoulli polynomials. $\zeta(1)$ remains truly divergent but the $n = 2$ contribution to the effective action vanishes anyway, because the respective trace of the A-fields equals zero, see Appendix B.

2. Fermions

The fermionic integrals of interest are

$$I_n^f(a) = T^{3-n} \left(J_n^f(a) + (-1)^{n+1} J_n^f(-a) \right) \tag{A6}$$

where $a \equiv \beta\mu$ and

$$J_n^f(a) = \int_0^{\infty} dx \, x^2 \, \frac{d^n f_{\mathrm{FD}}(x-a)}{dx^n} \ . \tag{A7}$$

Expanding the Fermi-Dirac distribution as

$$f_{\mathrm{FD}}(x-a) = \frac{1}{e^{x-a}+1} = \sum_{m=1}^{\infty} (-1)^{m-1} e^{-m(x-a)} \ ,$$

and interchanging the order of summation and integration, after performing the integral (A2), we obtain

$$J_n^f(a) = 2(-1)^n \sum_{m=1}^{\infty} (-1)^{m-1} \frac{e^{ma}}{m^{3-n}} \ . \tag{A8}$$

Expanding the exponential function and interchanging the order of the two summations in Eq. (A8), we find

$$J_n^f(a) = 2(-1)^n \sum_{l=0}^{\infty} \frac{a^l}{l!}\, \eta(3-n-l) \ , \tag{A9}$$

where $\eta(s)$ is the alternating zeta function defined as

$$\eta(s) \stackrel{\text{def}}{=} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s} = (1-2^{1-s})\, \zeta(s) \ , \qquad \mathrm{Re}s > 1 \ . \tag{A10}$$

Inserting the series (A9) into Eq. (A6) one gets

$$I_n^f(a) = 2T^{3-n} \sum_{l=0}^{\infty} \frac{a^l}{l!}\, \eta(3-n-l) \Big[(-1)^n - (-1)^l\Big] \ . \tag{A11}$$

As seen in Eq. (A11), the argument of η is always an even number for non-vanishing terms. Since $\eta(-2k) = 0$ for $k = 1, 2, ...$ the series in Eq. (A11) terminates. Then, one observes that $I_n^f(a) = 0$ for $n \geq 4$, while

$$\begin{aligned}
I_0^f(a) &= T^3 \left(4a\, \eta(2) + \frac{2}{3} a^3 \eta(0)\right) = T^3 \left(\frac{\pi^2}{3}\, a + \frac{1}{3}\, a^3\right) \ , \\
I_1^f(a) &= T^2 \left(-4\, \eta(2) - 2a^2\, \eta(0)\right) = -T^2 \left(\frac{\pi^2}{3} + a^2\right) \ , \\
I_2^f(a) &= 4aT\, \eta(0) = 2aT \ , \\
I_3^f(a) &= -4\, \eta(0) = -2 \ ,
\end{aligned} \tag{A12}$$

where $\eta(2) = \pi^2/12$ and $\eta(0) = 1/2$.

APPENDIX B: ADJOINT REPRESENTATION TRACES

To compute the effective action of Eq. (5.6) and Eq. (5.7) one needs to evaluate traces in the fundamental and adjoint representations. We present here some useful formulas which relate the traces of the fundamental and adjoint representations.

First, it is easy to prove that the total symmetric traces of an odd number of adjoint generators vanish. In order to prove this, note that

$$\mathrm{Tr}[\mathcal{A}_0^m] = \mathrm{Tr}[(\mathcal{A}_0^m)^T] \ , \tag{B1}$$

where the superscript T denotes transposition. The adjoint representation of $SU(N_c)$ is real, and the generators obey $T_a^T = -T_a$. Therefore,

$$\mathrm{Tr}[\mathcal{A}_0^m] = (-1)^m \mathrm{Tr}[\mathcal{A}_0^m] \ . \tag{B2}$$

Thus, the trace vanishes for odd m.

Two other useful formulas are:

$$\mathrm{Tr}\mathcal{A}_0^2 = 2N_c \mathrm{Tr} A_0^2 \ , \tag{B3}$$
$$\mathrm{Tr}\mathcal{A}_0^4 = 6(\mathrm{Tr}A_0^2)^2 + 2N_c \mathrm{Tr}A_0^4 \ . \tag{B4}$$

For $N_c = 2$ and $N_c = 3$, one also has

$$\mathrm{Tr}A_0^4 = \frac{1}{2}(\mathrm{Tr}A_0^2)^2 \ . \tag{B5}$$

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